

## OPTIMALITY IN A BISECTORIAL MODEL FOR ECONOMIC GROWTH

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**Abstract:** This paper continues the study proceeded in [2], where we have given two analytical solutions for the bisectorial model for economic growth introduced by Uzawa in [3] and [4], as an extension of the Solow-Swan neoclassical growth model considered in [1]. A locally stable solution for the evolution equation that describes the dynamics of the capital was also provided.

The theory of control theory and the qualitative study of the optimum led to an effective solution of the problems raised in the mathematical modelling of the economic growth. In this paper, a model proposed by Uzawa and Srinivasan is discussed. The path of optimal growth, which maximizes the integral of the consumption path, is given. The study of multisectorial economic models shows the fact that the convex technological frame of growth models allows to emphasize some remarkable properties for the trajectories that satisfy the optimality conditions, considering also the resource theory.

### 1. PRELIMINARIES AND NOTATIONS

We will consider an economy that produces a commodity for consumption and a commodity for investment by means of capital and labour. The bisectorial model described by Uzawa in [3] and [4] extends the Solow-Swan neoclassical growth model with a less restrictive relation between capital and income. The Uzawa model is characterized by two inputs and two outputs, one of the latest being also an input. The main result of Uzawa model is the fact that the uniqueness of the equilibrium on the production factor market and the stability of the steady-state growth path depend on the relative factor-intensities of the two sectors and attendant savings hypotheses. The stability is assured if the sector of consumer goods is more capital-intensive than the sector of investment goods. Uzawa and Srinivasan tried to give a solution without such a radical capital-intensity assumption and, hence, studied the optimal growth path, specifically the growth path that maximizes the integral of the consumption path.

The aim is to give the optimal plan in the repartition of the production forces existing on the market between the two sectors, which means to describe an allocation plan toward the two sectors which should maximize the per capita consumption flow, on a given time lag. In this specific case presented in our paper, the planning horizon is infinite.

Thus, the problem is to determine the optimal solution for

$$\max \int_0^{\infty} y_c(t) e^{-\alpha t} dt, \quad (1.1)$$

with the constraints

$$\begin{aligned} \frac{dk}{dt} &= y_i - \lambda k, \quad k(0) = k_0 \\ y_c &= \xi_c f_c(k_c), \quad y_i = \xi_i f_i(k_i) \\ \xi_c + \xi_i &= 1, \quad \xi_c k_c + \xi_i k_i = k \\ \xi_c, k_c, \xi_i, k_i &\geq 0, \end{aligned}$$

where the notations are those presented in [1] and [2], such as the capital stock per capita  $k$ , the production function per capita  $f$ , the income per capita  $y$ , the labor level per capita  $\xi$  and the labor growth rate  $\lambda$ . The index  $c$  or  $i$  indicates if the consumption sector or the

investments sector is concerned. We consider also the relations concerning the dynamic of the capital  $k'(t) = y(t) - \lambda k(t)$ , for all  $t \geq 0$ , the equilibrium in the labor market, described by  $\xi_c(t) + \xi_i(t) = 1$ , for all  $t \geq 0$ , as well as the equilibrium in the capital market, given by the relation  $k_c(t)\xi_c(t) + k_i(t)\xi_i(t) = k(t)$ , for all  $t \geq 0$ .

**Remark.** The consumption per capita is the output of the sector dedicated to consumptions goods. The factor  $\delta$  gives the time preference of the investor and the discount rate. The problem is well defined, as the objective integral is bounded on the set of all admissible consumption paths.

The intensive production functions in the Uzawa bisectorial model can also be written as

$$y_c = \frac{k - k_i}{k_c - k_i} f_c(k_c), \text{ respectively } y_i = \frac{k_c - k}{k_c - k_i} f_i(k_i). \quad (1.2)$$

The functions  $k_c = k_c(\rho)$  and  $k_i = k_i(\rho)$ , having the properties

$$k_c = \frac{(f_c)'}{f_c''} > 0 \text{ and } k_i = \frac{(f_i)'}{f_i''} > 0$$

are the margins of the equilibrium of the market of production factors. We have thus  $\rho \in [\rho_{\min}, \rho_{\max}]$ , where  $\rho$  denotes the wage-profit ratio. If the consumer goods sector is more capital-intensive, which means  $k_c(\rho) > k_i(\rho)$ ,  $\forall \rho$ , we have  $\rho_{\min} = \rho_c(k)$  and  $\rho_{\max} = \rho_i(k)$ , and if the investment goods sector is more capital-intensive, that is  $k_c(\rho) \leq k_i(\rho)$ ,  $\forall \rho$ , we have  $\rho_{\min} = \rho_i(k)$  and  $\rho_{\max} = \rho_c(k)$ .

## 2. THE MAIN RESULTS

In this section we will study the optimality conditions and we will characterize the optimal growth paths.

### 2.1. Optimality conditions

Let us consider two cases, according to the property of being more capital-intensive of the two sectors: the sector of the consumer goods or the sector investment goods. We will rewrite the integral of the consumption path (1.1) according to the relations (1.2). We will consider  $k$  the state variable and  $\rho$  the control variable

**Case 1.** Let  $k_c(\rho) > k_i(\rho)$ , for all admissible values of  $\rho$ . We have  $\rho \in [\rho_c(k), \rho_i(k)]$ . The problem can be written as

$$\max \int_0^{\infty} \frac{k - k_i}{k_c - k_i} f_c(k_c) e^{-\delta t} dt \quad (2.1)$$

with the hypothesis

$$\frac{dk}{dt} = \frac{k_c - k}{k_c - k_i} f_i(k_i) - \lambda k, \quad k(0) = k_0.$$

Let us consider the Hamiltonian  $H$  with the costate variable denoted by  $x$

$$H = \frac{k - k_i}{k_c - k_i} f_c(k_c) + x \left[ \frac{k_c - k}{k_c - k_i} f_i(k_i) - \lambda k \right]. \quad (2.2)$$

We obtain the maximum conditions from

$$\frac{dH}{d\rho} = (x f_i' - f_c') \left( \frac{dk_i}{d\rho} \lambda_c \frac{\rho + k_i}{k_c - k_i} + \frac{dk_c}{d\rho} \lambda_i \frac{\rho + k_c}{k_c - k_i} \right) = 0 \quad (2.3)$$

and, further, the value of the costate variable  $x = \frac{f_c}{f_i}$ , which is the price  $p$  of the goods for investment. It follows that

$$\frac{dx}{dt} = x(\lambda + \delta) + \frac{xf_i(k_i) - f_c(k_c)}{k_c - k_i} = x(\lambda + \delta - f_i').$$

As  $x = p(\rho)$  and  $\frac{dx}{dt} = \frac{dp}{d\rho} \cdot \frac{d\rho}{dt}$  we obtain  $\frac{d\rho}{dt} = p \cdot \frac{\lambda + \delta - f_i'}{\frac{dp}{d\rho}}$ . Hence, following relation holds

$$\frac{d\rho}{dt} = p \cdot \frac{\lambda + \delta - f_i'}{\frac{1}{\rho + k_i} - \frac{1}{\rho + k_c}}. \quad (2.4)$$

From the relation  $\frac{dH}{dx} = \frac{dk}{dt}$  we obtain

$$\frac{dk}{dt} = \frac{k_c - k}{k_c - k_i} f_i(k_i) - \lambda k. \quad (2.5)$$

Relations (2.4) and (2.5) are the differential equations of the model.

On the first hand, we will consider  $\frac{d\rho}{dt} = 0$ , which implies  $\lambda + \delta = f_i'(k_i(\rho))$ , and, hence, the equilibrium state is given by  $\rho^* = k_i^{-1}(f_i'^{-1}(\lambda + \delta))$ . The dynamics is given by

$$\frac{d \frac{d\rho}{dt}}{d\rho}(\rho^*) = -\frac{f_i''}{\frac{1}{p} \frac{dp}{d\rho}} > 0,$$

which implies that if  $\rho > \rho^*$  we obtain an appreciation and if  $\rho < \rho^*$  a depreciation.

On the other hand, if we consider  $\frac{dk}{dt} = 0$ , as  $\frac{k_c - k}{k_c - k_i} f_i(k_i) = \lambda k$ , we obtain

$$k(\rho) = \frac{f_i(k_i)k_c}{\lambda(k_c - k_i) + f_i(k_i)} \text{ and } k_i(\rho) \leq k(\rho) \leq k_c(\rho), \quad \forall \rho \leq \rho^*.$$

We also have

$$\frac{d \frac{dk}{dt}}{dk} = -\frac{f_i}{k_c - k_i} - \lambda < 0.$$

**Case 2.** Let  $k_c(\rho) < k_i(\rho)$ , for all admissible values of  $\rho$ . We have  $\rho \in [\rho_i(k), \rho_c(k)]$ . The problem is given by

$$\max \int_0^{\infty} \frac{k - k_i}{k_c - k_i} f_c(k_c) e^{-\delta t} dt$$

with the hypothesis

$$\frac{dk}{dt} = \frac{k_c - k}{k_c - k_i} f_i(k_i) - \lambda k, \quad k(0) = k_0.$$

The Hamiltonian and the maximum conditions are given by relations (2.2) and (2.3). In this case, the differential equations of the model are

$$\frac{d\rho}{dt} = \frac{\lambda + \delta - f_i'}{\frac{1}{\rho + k_i} - \frac{1}{\rho + k_c}} \quad (2.6)$$

and, respectively, (2.5).

If  $\frac{d\rho}{dt} = 0$ , the equilibrium state is given by  $\rho^* = k_i^{-1}(f_i'^{-1}(\lambda + \delta))$ . The dynamics is given by

$$\frac{d}{d\rho} \frac{d\rho}{dt}(\rho^*) = -\frac{f_i''}{\frac{1}{p} \frac{d\rho}{d\rho}} < 0,$$

and, hence, the case of stability is obtained.

If  $\frac{dk}{dt} = 0$  we have  $k_c - k = \frac{\lambda k}{f_i(k_i)}(k_c - k_i)$  and  $k(\rho) > k_c(\rho)$ , for all  $\rho$ . We also have

$$\frac{d}{dk} \frac{dk}{dt} = -\frac{f_i}{k_c - k_i} - \lambda > 0.$$

## 2.2. Characterization of the optimal growth paths

In what follows we will introduce a new command parameter. The weight or share of the sector due to consumption in the general income per capita is denoted  $c$ ,  $c \in [0,1]$ , and the propensity to savings or the weight of the investment sector in the aggregate income is denoted by  $s = 1 - c$ ,  $s \in [0,1]$ .

We have following cases: if  $s = 0$ , the economy will be specialized in the production of consumption goods, and if  $s = 1$ , all incomes are reinvested, respectively are used to obtain new production means.

In what follows, Case 2 presented in paragraph 2.1 will be formulated in order to determine

$$\max \int_0^{\infty} [1 - s(t)]y(t)e^{-\delta t} dt \quad (2.7)$$

with the hypothesis

$$\frac{dk(t)}{dt} = \frac{s(t)}{p(t)}y(t) - \lambda k(t), \quad k(0) = k_0$$

$$s(t) \in [0,1].$$

We will denote by  $x = x(t)$  the costate variable attached to the state equation. By  $l_c, l_i$  we denote the labour levels per capita in the consumption sector, respectively in the investment sector.

**Theorem 2.1.** *Let  $(k(t), k_c(t), k_i(t), l_c(t), l_i(t))$  be the optimal growth path expressed per capita. There exists a function  $g = g(t)$  such that following relations hold:*

- (i)  $g' = [\lambda + \delta - f_i'(k_i)]g$ ,  $g = p$ ;
- (ii)  $g' = [\lambda + \delta - f_i'(k)]g$ ,  $g > p$ ;
- (iii)  $g' = (\lambda + \delta)g - f_c'(k)$ ,  $g < p$ .

**Proof.** Let us consider the function  $g(t) = x(t)e^{\delta t}$ . The Hamiltonian is given by the relation

$$H(t, g(t), k(t), s(t)) = [1 - s(t)]y(t) + \left( \frac{s(t)}{p(t)}y(t) - \lambda k(t) \right) g(t).$$

At every moment  $t$  the optimal command maximizes the Hamiltonian on  $[0, 1]$ , which implies  $s \in (0, 1)$  if  $\frac{\partial H}{\partial s} = 0$ ,  $s = 0$  if  $\frac{\partial H}{\partial s} < 0$  and  $s = 1$  if  $\frac{\partial H}{\partial s} > 0$ .

Let us fix  $t$  and denote  $H(s) = H(s, t, g(t), k(t))$ . The costate variable  $x$  satisfy the differential equation  $x' = -\frac{\partial H}{\partial k}$ , which implies

$$g' = (\lambda + \delta)g - \left[ \frac{\partial(1-s)y}{\partial k} + \frac{\partial\left(\frac{s}{p}y\right)}{\partial k} \right].$$

According to the relations that define the production function per capita, given [2], we have

$$\frac{\partial y_c}{\partial \rho} = -\rho \frac{\partial y_i}{\partial \rho}.$$

As following relation holds

$$\frac{\partial H}{\partial s} = \frac{\partial y_c}{\partial \rho} \frac{\partial \rho}{\partial s} + \frac{\partial y_i}{\partial \rho} \frac{\partial \rho}{\partial s} g, \quad \frac{\partial y_i}{\partial \rho} \frac{\partial \rho}{\partial s} > 0,$$

we obtain that  $s \in (0, 1)$  if  $g - \rho = 0$ ,  $s = 0$  if  $g - \rho > 0$  and  $s = 1$  if  $g - \rho < 0$ .

As we have that following relations are true

$$\frac{\partial(1-s)y}{\partial k} = \left( \frac{k_c + \rho}{k_c - k_i} f'_c - \frac{\partial y_i}{\partial \rho} \frac{\partial \rho}{\partial s} \right) \rho,$$

$$\frac{\partial\left(\frac{s}{p}y\right)}{\partial k} = \frac{k_i + \rho}{k_c - k_i} f'_i + \frac{\partial y_i}{\partial \rho} \frac{\partial \rho}{\partial s}$$

we obtain that the necessary optimality conditions lead to the equations

$$g' = [\lambda + \delta - f'_i(k_i)]g \quad \text{and} \quad g' = [\lambda + \delta - f'_c(k)]g$$

if the economy is focused on investments, respectively

$$g' = (\lambda + \delta)g - f'_c(k)$$

if the production of consumption goods prevails.

Hence, the proof is completed.

In what follows we will prove that, together with a transversality condition, the necessary optimality conditions are also sufficient. We will denote

$$u = \max(gf'_i, f'_c) \quad \text{and} \quad v = \max(g(f_i - k_i f'_i), f_c - k_c f'_c).$$

Then  $gf'_i < u$  implies  $l_i = 0$  and  $k_i = k_i(\rho_c(k))$ ;  $f'_c < u$  implies  $l_c = 0$  and  $k_c = k_c(\rho_i(k))$ ;  $g(f_i - k_i f'_i) < v$  implies  $l_i = 0$  and  $k_i = k_i(\rho_c(k))$  and  $f_c - k_c f'_c < v$  implies  $l_c = 0$  and  $k_c = k_c(\rho_i(k))$ . Also, the previous equations can be reduced to

$$g' = (\lambda + \delta)g - u.$$

**Theorem 2.2.** *If  $(k(t), k_c(t), k_i(t), l_c(t), l_i(t))$  is a growth path for which there exists a function  $g = g(t)$  such that the necessary optimality conditions and a transversality condition  $\lim_{t \rightarrow \infty} k(t)g(t)e^{-\delta t} = 0$  are satisfied, then this path is optimal.*

**Proof.** Let us consider a growth path  $(K(t), K_c(t), K_i(t), L_c(t), L_i(t))$  which verifies necessary optimality conditions and the transversality condition. We will prove that this path is optimal.

Let us consider another growth path  $(\bar{K}(t), \bar{K}_c(t), \bar{K}_i(t), \bar{L}_c(t), \bar{L}_i(t))$ . We obtain

$$\begin{aligned} & \int_0^{\infty} \frac{1}{L} F_c(K_c, L_c) e^{-\delta t} dt - \int_0^{\infty} \frac{1}{L} F_c(\bar{K}_c, \bar{L}_c) e^{-\delta t} dt = \\ &= \frac{1}{L} \int_0^{\infty} [F_c(K_c, L_c) - F_c(\bar{K}_c, \bar{L}_c) - u(K_c - \bar{K}_c) - v(L_c - \bar{L}_c)] e^{-\delta t} dt + \\ &+ \frac{1}{L} \int_0^{\infty} [F_i(K_i, L_i) g - F_i(\bar{K}_i, \bar{L}_i) g - u(K_i - \bar{K}_i) - v(L_i - \bar{L}_i)] e^{-\delta t} dt + \\ &+ \frac{1}{L} \int_0^{\infty} [(u - \lambda g)(K - \bar{K}) - (K' - \bar{K}') g] e^{-\delta t} dt. \end{aligned}$$

According to the concavity of the production function  $F_i$ , and, successively to the propensity toward one sector or the other, following relations hold

$$\begin{aligned} & [F_i(K_i, L_i) - F_i(\bar{K}_i, \bar{L}_i)] g - u(K_i - \bar{K}_i) - v(L_i - \bar{L}_i) \geq 0 \\ & F_c(K_c, L_c) - F_c(\bar{K}_c, \bar{L}_c) - u(K_c - \bar{K}_c) - v(L_c - \bar{L}_c) \geq 0 \end{aligned}$$

From the equality  $u = (\lambda + \delta)g - g'$  and from the transversality condition, we have

$$\begin{aligned} & \int_0^{\infty} \frac{1}{L} F_c(K_c, L_c) e^{-\delta t} dt - \int_0^{\infty} \frac{1}{L} F_c(\bar{K}_c, \bar{L}_c) e^{-\delta t} dt \geq \\ & \geq \int_0^{\infty} \frac{1}{L} [(u - \lambda g)(K - \bar{K}) - (K' - \bar{K}') g] e^{-\delta t} dt = \\ &= \int_0^{\infty} \frac{1}{L} [(\lambda g + \delta g - g' - \lambda g)(k - \bar{k}) - (-\lambda k + k' + \lambda \bar{k} - \bar{k}') g] e^{-\delta t} dt = \\ &= \lim_{T \rightarrow \infty} [\bar{k}(T) - k(T)] g(T) e^{-\delta T} \geq 0, \end{aligned}$$

which implies that the growth path  $(K(t), K_c(t), K_i(t), L_c(t), L_i(t))$  is optimal and which ends the proof.

**Remark.** If the optimum problem (2.7) admits an optimal solution, then this is uniquely determined.

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