

## SOME CONSIDERATIONS ABOUT THE DYNAMICS OF THE REINFORCED CONCRETE BRIDGES MODELED AS A RIGID SOLID WITH VISCOUS-ELASTIC BEARINGS

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**Keywords:** linear modeling, natural frequencies, eigenvalues, elastic bearings, reinforced concrete bridges

**Abstract:** The article proposes an approach of six degrees dynamic model of a rigid-solid with some types of symmetries. These symmetries lead to simplified mathematical models, which are more easily to solve. If the rigid-solid is jointed of the structure by four elastic links, the mathematical model becomes still simple and the vibrations are decoupled into four subsystems of movements: side slipping and rolling, forward motion and pitching, lifting motion, gyration. There is also a case study: the modal analysis of a bridge with the total length of 200 meters, 13.2 meters width and 2.5 meters height, modeled as a rigid solid elastically beared.

### 1. INTRODUCTION. MATHEMATICAL MODELING OF THE RIGID SOLID

The mathematical modeling uses the physical model of the rigid solid with six degrees of freedom (6DOF) with a finite number of viscous-elastic bearings. Dimensional and inertial characteristics of the rigid solid and rheological characteristics of the bearings (stiffness and damping) can be experimentally determined by direct measurements and by static and/or dynamic testing. The differential equations of the movements of linear vibration of the rigid solid with viscous-elastic bearings on three orthogonal directions are coupled by stiffness and damping coefficients [9]. The system of the equations can be write as follows

$$\underline{A}\ddot{\underline{q}} + \underline{B}\dot{\underline{q}} + \underline{C}\underline{q} = \underline{f} , \quad (1)$$

where  $\underline{A}$  is the inertia matrix (masses, statical moments, moments of inertia)

$\underline{B}$  - viscous damping matrix (damping coefficients)

$\underline{C}$  - elasticity matrix (stiffness coefficients)

$\underline{q} / \dot{\underline{q}} / \ddot{\underline{q}}$  - generalized displacements / velocities / accelerations vector

$\underline{f}$  - generalized forces vector

If the damping coefficients are small (rigid solid with quasielastic bearings), the differential equations system becomes:

$$\underline{A}\ddot{\underline{q}} + \underline{C}\underline{q} = \underline{f} \quad (2)$$

In order to establish the natural pulsation / frequencies and the eigenvalues, we consider the rigid solid elastically beared with linear stiffness and no perturbations. The system of differential equations is now:

$$\underline{A}\ddot{\underline{q}} + \underline{C}\underline{q} = \underline{0} , \quad (3)$$

where  $\underline{0}$  is the null vector (where all coefficients are zero).

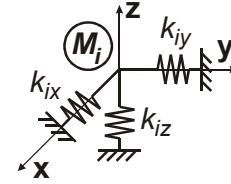
If the Cartesian coordinates axis system is central and principal, the quadratic  $6 \times 6$  inertia matrix becomes diagonal

$$\underline{A} = \text{DIAG}[m, m, m, J_x, J_y, J_z] , \quad (4)$$

where  $m$  is the rigid solid mass and  $J_x, J_y, J_z$  are the principal inertia moments.

## 2. THE MODAL ANALYSIS OF THE RIGID SOLID WITH STRUCTURAL SYMMETRIES

We consider that the rigid solid is symmetric (mass distribution, geometrical configuration, bearings disposal) and the coordinate system is central and principal, thus the inertia matrix is diagonal. If the elastic bearing system of the rigid solid is composed from  $n$  supports with triorthogonal stiffness ( $k_{ix}, k_{iy}, k_{iz}$ ) like in **figure 1** with the position done by the coordinates  $M_i(x_i, y_i, z_i) \quad i = \overline{1, n}$ , the elasticity matrix becomes:



**Fig. 1 Elastic triorthogonal bearing**

$$\underline{C} = \begin{bmatrix} \sum k_{ix} & 0 & 0 & 0 & \sum k_{ix}z_i & 0 \\ 0 & \sum k_{iy} & 0 & -\sum k_{iy}z_i & 0 & 0 \\ 0 & 0 & \sum k_{iz} & 0 & 0 & 0 \\ 0 & -\sum k_{iy}z_i & 0 & \sum (k_{iy}z_i^2 + k_{iz}y_i^2) & 0 & 0 \\ \sum k_{ix}z_i & 0 & 0 & 0 & \sum (k_{iz}x_i^2 + k_{ix}z_i^2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sum (k_{ix}y_i^2 + k_{iy}x_i^2) \end{bmatrix} \quad (5)$$

As the inertia matrix is diagonal, the coefficients outside the main diagonal of the elasticity matrix  $\underline{C}$  are the coupling terms of the equations of the system (3). Because there are only four non-zero stiffness coefficients ( $c_{15} \equiv c_{51}$  and  $c_{24} \equiv c_{42}$ ), the free movements of the rigid solid are decoupled into four subsystems with coupled vibrations. The subsystems with coupled motion equations are as follows:

a) subsystem  $(X, \varphi_y)$  - side slip movement coupled with rolling movement

$$\begin{cases} m\ddot{X} + X\sum k_{ix} + \varphi_y\sum k_{ix}z_i = 0 \\ J_y\ddot{\varphi}_y + X\sum k_{ix}z_i + \varphi_y\sum (k_{iz}x_i^2 + k_{ix}z_i^2) = 0 \end{cases} \quad (6)$$

or the canonical form of the system

$$\begin{cases} \ddot{X} + p_X^2 X + \alpha_1 \varphi_y = 0 \\ \ddot{\varphi}_y + \alpha_2 X + p_{\varphi_y}^2 \varphi_y = 0 \end{cases} \quad (6a)$$

b) subsystem  $(Y, \varphi_x)$  - forward-back movement coupled with pitch movement

$$\begin{cases} m\ddot{Y} + Y\sum k_{iy} - \varphi_x\sum k_{iy}z_i = 0 \\ J_x\ddot{\varphi}_x - Y\sum k_{iy}z_i + \varphi_x\sum (k_{iy}z_i^2 + k_{iz}y_i^2) = 0 \end{cases} \quad (7)$$

or the canonical form of the system

$$\begin{cases} \ddot{Y} + p_Y^2 Y + \beta_1 \varphi_x = 0 \\ \ddot{\varphi}_x + \beta_2 Y + p_{\varphi_x}^2 \varphi_x = 0 \end{cases} \quad (7a)$$

c) subsystem  $(Z)$  - up-down movement

$$m\ddot{Z} + Z\sum k_{iz} = 0 \quad \text{or} \quad \ddot{Z} + p_Z^2 Z = 0 \quad (8)$$

d) subsystem  $(\varphi_z)$  - turning movement (gyration)

$$J_z\ddot{\varphi}_z + \varphi_z\sum (k_{ix}y_i^2 + k_{iy}x_i^2) = 0 \quad \text{or} \quad \ddot{\varphi}_z + p_{\varphi_z}^2 \varphi_z = 0 \quad (9)$$

The notations from the equations (6) to (9) are the next:

► the pulsations of the no coupled movements

$$p_X = \sqrt{\frac{\sum k_{ix}}{m}} \quad (10)$$

$$p_Y = \sqrt{\frac{\sum k_{iy}}{m}} \quad (11)$$

$$p_Z = \sqrt{\frac{\sum k_{iz}}{m}} \quad (12)$$

$$p_{\varphi_X} = \sqrt{\frac{\sum (k_{iy}z_i^2 + k_{iz}y_i^2)}{J_X}} \quad (13)$$

$$p_{\varphi_Y} = \sqrt{\frac{\sum (k_{iz}x_i^2 + k_{ix}z_i^2)}{J_Y}} \quad (14)$$

$$p_{\varphi_Z} = \sqrt{\frac{\sum (k_{ix}y_i^2 + k_{iy}x_i^2)}{J_Z}} \quad (15)$$

► the dynamic coupling terms for the  $(X, \varphi_Y)$  and  $(Y, \varphi_X)$  subsystems

$$\begin{cases} \alpha_1 = \frac{1}{m} \sum k_{ix}z_i \\ \alpha_2 = \frac{1}{J_Y} \sum k_{ix}z_i \end{cases} \quad (16)$$

$$\begin{cases} \beta_1 = -\frac{1}{m} \sum k_{iy}z_i \\ \beta_2 = -\frac{1}{J_X} \sum k_{iy}z_i \end{cases} \quad (17)$$

Considering the above notations, the natural pulsations and the eigenvalues of the decoupled subsystems can be determinate with the next calculus formula:

a) for the subsystem  $(X, \varphi_Y)$

$$p_{1,2} = \sqrt{\frac{1}{2} \left[ p_X^2 + p_{\varphi_Y}^2 \mp \sqrt{(p_X^2 - p_{\varphi_Y}^2)^2 + 4\alpha_1\alpha_2} \right]} \quad (18)$$

$$\mu_{1,2} = -\frac{1}{2\alpha_1} \left[ p_X^2 + p_{\varphi_Y}^2 \pm \sqrt{(p_X^2 - p_{\varphi_Y}^2)^2 + 4\alpha_1\alpha_2} \right] \quad (19)$$

b) for the subsystem  $(Y, \varphi_X)$

$$p_{3,4} = \sqrt{\frac{1}{2} \left[ p_Y^2 + p_{\varphi_X}^2 \mp \sqrt{(p_Y^2 - p_{\varphi_X}^2)^2 + 4\beta_1\beta_2} \right]} \quad (20)$$

$$\mu_{3,4} = -\frac{1}{2\beta_1} \left[ p_Y^2 + p_{\varphi_X}^2 \pm \sqrt{(p_Y^2 - p_{\varphi_X}^2)^2 + 4\beta_1\beta_2} \right] \quad (21)$$

### 3. CASE STUDY – MODAL ANALYSIS OF A BRIDGE MADE FROM REINFORCED CONCRETE “U” BEAMS

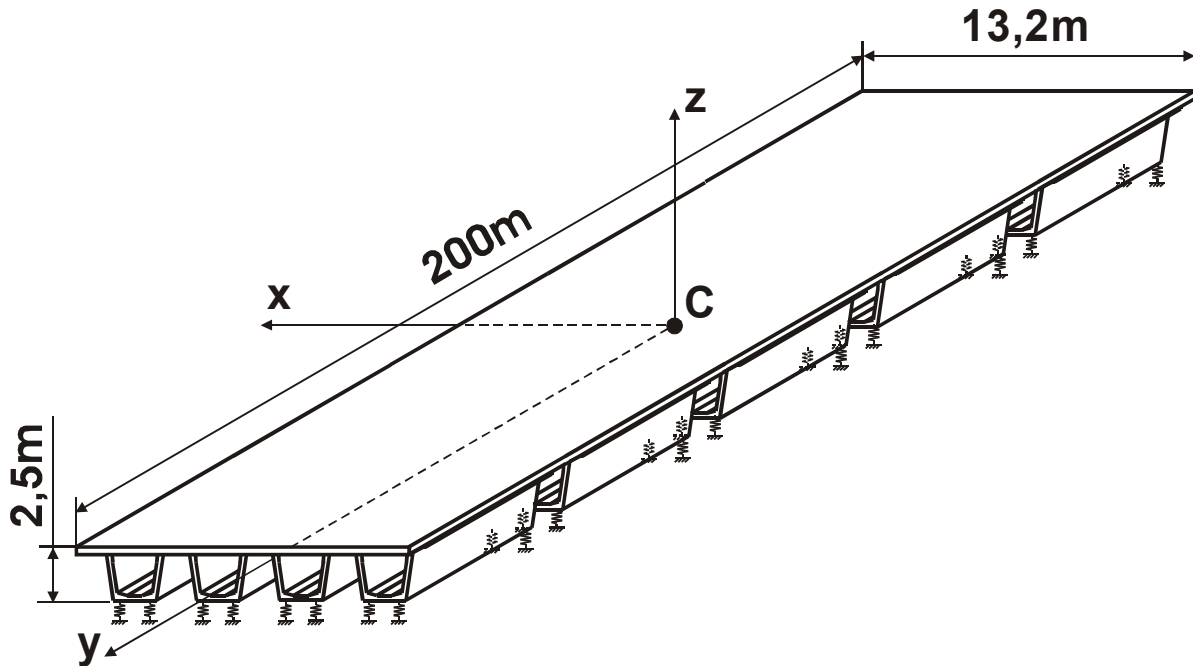


Fig. 2 The model of the bridge beared on eighty steel plates reinforced neoprene supports

Figure 2 shows the simplified model of a bridge made from twenty reinforced concrete beams jointed through a 300 mm thickness reinforced concrete plate. Each beam is beared on the piles of the bridge through four identically viscous-elastic supports made from neoprene (figure 3); the supports, having the sizes  $500 \times 400 \times 81$  [mm], are reinforced trough six steel plates of 5 mm thickness.

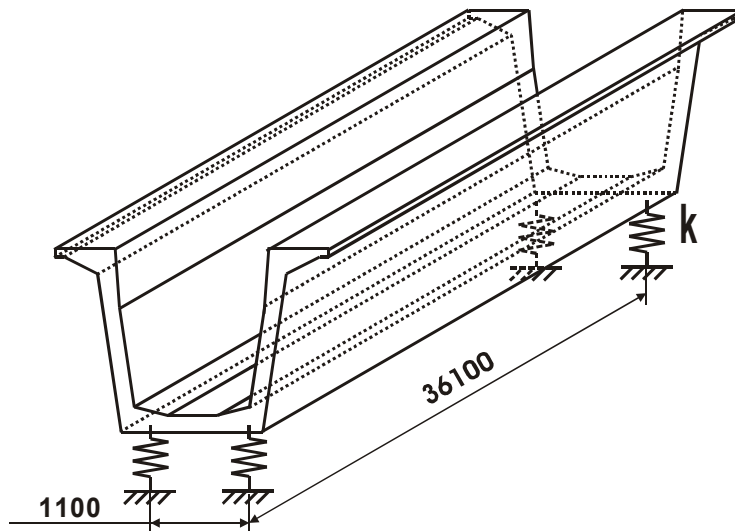


Fig. 3 The model of the “U” beam beared on neoprene supports

In order to calculate the natural pulsations and frequencies and the eigenvalues of the bridge modeled as in the figure 2, the main characteristics (inertial, dimensional, stiffness) have been calculated and/or measured and are the next:

■Dimensions (as in detailed engineering drawings and measured):

•for “U” beams:  $37100 \times 1700 / 3280 \times 2200$  length $\times$ width $\times$ height [mm]

•for the bridge:  $200000 \times 13300 \times 2500$  length $\times$ width $\times$ height [mm]

■Masses and inertia (calculated):

$$m = 4960000 \text{ kg}$$

$$J_{xy} = J_{yz} = J_{zx} = 0$$

$$J_x = 16,025 \times 10^9 \text{ kgm}^2$$

$$J_y = 73,270 \times 10^6 \text{ kgm}^2$$

$$J_z = 16,092 \times 10^9 \text{ kgm}^2$$

■Stiffness of the neoprene bearings (measured):

$$k_{ix} \equiv k_x = 3,15 \times 10^6 \text{ N/m} \quad i = \overline{1,80}$$

$$k_{iy} \equiv k_y = 3,15 \times 10^6 \text{ N/m} \quad i = \overline{1,80}$$

$$k_{iz} \equiv k_z = 650 \times 10^6 \text{ N/m} \quad i = \overline{1,80}$$

■Position of the mass center **C** against the neoprene bearings (calculated):

$$h = 1454,4 \text{ mm}$$

■Positions of the neoprene bearings against the centered coordinate system **Cxyz** (as in detailed engineering drawings) – see **table 1**.

**Table 1 –positions of the neoprene bearings**

Bearing and coordinates [m]															
i	x <sub>i</sub>	y <sub>i</sub>	z <sub>i</sub>	i	x <sub>i</sub>	y <sub>i</sub>	z <sub>i</sub>	i	x <sub>i</sub>	y <sub>i</sub>	z <sub>i</sub>	i	x <sub>i</sub>	y <sub>i</sub>	z <sub>i</sub>
1	-5,5	-98,05	-1,45	21	1,1	-58,05	-1,45	41	-5,5	18,05	-1,45	61	1,1	58,05	-1,45
2	-4,4	-98,05	-1,45	22	2,2	-58,05	-1,45	42	-4,4	18,05	-1,45	62	2,2	58,05	-1,45
3	-2,2	-98,05	-1,45	23	4,4	-58,05	-1,45	43	-2,2	18,05	-1,45	63	4,4	58,05	-1,45
4	-1,1	-98,05	-1,45	24	5,5	-58,05	-1,45	44	-1,1	18,05	-1,45	64	5,5	58,05	-1,45
5	1,1	-98,05	-1,45	25	-5,5	-21,95	-1,45	45	1,1	18,05	-1,45	65	-5,5	61,95	-1,45
6	2,2	-98,05	-1,45	26	-4,4	-21,95	-1,45	46	2,2	18,05	-1,45	66	-4,4	61,95	-1,45
7	4,4	-98,05	-1,45	27	-2,2	-21,95	-1,45	47	4,4	18,05	-1,45	67	-2,2	61,95	-1,45
8	5,5	-98,05	-1,45	28	-1,1	-21,95	-1,45	48	5,5	18,05	-1,45	68	-1,1	61,95	-1,45
9	-5,5	-61,95	-1,45	29	1,1	-21,95	-1,45	49	-5,5	21,95	-1,45	69	1,1	61,95	-1,45
10	-4,4	-61,95	-1,45	30	2,2	-21,95	-1,45	50	-4,4	21,95	-1,45	70	2,2	61,95	-1,45
11	-2,2	-61,95	-1,45	31	4,4	-21,95	-1,45	51	-2,2	21,95	-1,45	71	4,4	61,95	-1,45
12	-1,1	-61,95	-1,45	32	5,5	-21,95	-1,45	52	-1,1	21,95	-1,45	72	5,5	61,95	-1,45
13	1,1	-61,95	-1,45	33	-5,5	-18,05	-1,45	53	1,1	21,95	-1,45	73	-5,5	98,05	-1,45
14	2,2	-61,95	-1,45	34	-4,4	-18,05	-1,45	54	2,2	21,95	-1,45	74	-4,4	98,05	-1,45
15	4,4	-61,95	-1,45	35	-2,2	-18,05	-1,45	55	4,4	21,95	-1,45	75	-2,2	98,05	-1,45
16	5,5	-61,95	-1,45	36	-1,1	-18,05	-1,45	56	5,5	21,95	-1,45	76	-1,1	98,05	-1,45
17	-5,5	-58,05	-1,45	37	1,1	-18,05	-1,45	57	-5,5	58,05	-1,45	77	1,1	98,05	-1,45
18	-4,4	-58,05	-1,45	38	2,2	-18,05	-1,45	58	-4,4	58,05	-1,45	78	2,2	98,05	-1,45
19	-2,2	-58,05	-1,45	39	4,4	-18,05	-1,45	59	-2,2	58,05	-1,45	79	4,4	98,05	-1,45
20	-1,1	-58,05	-1,45	40	5,5	18,05	-1,45	60	-1,1	58,05	-1,45	80	5,5	98,05	-1,45

Using the relations (10) to (15), the natural pulsations  $\rho$  and the natural frequencies  $f$  of the uncoupled vibrations for the six degrees of dynamic freedom are shown in the **table 2** (frequencies calculated from pulsations with  $f = \frac{\rho}{2\pi}$ ).

**Table 2 – Natural pulsations and frequencies (on the six degrees of dynamic freedom)**

Direction	X	Y	Z	$\varphi_x$	$\varphi_y$	$\varphi_z$
$p$ [rad/s]	7,13	7,13	103,39	106,96	99,41	7,44
$f$ [Hz]	1,13	1,13	16,30	17,02	15,82	1,18

The figures from **table 3** show the values of the natural pulsations and frequencies and of the eigenvalues for the decoupled subsystems (with coupled movements). As it can see, there are the same values for pulsations and frequencies like in **table 2**. That means, the movements inside the subsystems  $(X, \varphi_y)$  and  $(Y, \varphi_x)$  are very weak coupled, almost uncoupled.

**Table 3 - Natural pulsations / frequencies and the eigenvalues for the decoupled subsystems**

Subsystem	Pulsations	Frequencies	Eigenvalues
$(X, \varphi_y)$	$p_1 = 7,13 \text{ rad / s}$	$f_1 = 1,13 \text{ Hz}$	$\mu_1 = 0,000509 \text{ rad / m}$
	$p_2 = 99,41 \text{ rad / s}$	$f_2 = 15,82 \text{ Hz}$	$\mu_2 = -133,055 \text{ rad / m}$
$(Y, \varphi_x)$	$p_3 = 7,13 \text{ rad / s}$	$f_3 = 1,13 \text{ Hz}$	$\mu_3 = -0,000002 \text{ rad / m}$
	$p_4 = 106,96 \text{ rad / s}$	$f_4 = 17,02 \text{ Hz}$	$\mu_4 = 154,145 \text{ rad / m}$
$(Z)$	$p_5 = p_z = 103,39 \text{ rad / s}$	$f_5 = f_z = 16,30 \text{ Hz}$	-
$(\varphi_z)$	$p_6 = p_{\varphi_z} = 7,44 \text{ rad / s}$	$f_6 = f_{\varphi_z} = 1,18 \text{ Hz}$	-

#### 4. CONCLUSIONS

**a)** modeling a rigid solid with elastic or viscous-elastic bearings and symmetries (structural, inertial, bearings) lead to linear mathematical models more simple, with differential equations decoupled into subsystems easier to solve; in this case, we can highlight the influences of different kinds of characteristics (dimensions, masses, inertia, stiffness) on the dynamic parameters of the rigid solid (natural pulsations/frequencies, eigenvalues);

**b)** if the physical model of the rigid solid permits to chose a Cartesian coordinate system which is central and principal, then the differential equations of motion are coupled only by the coefficients outside of principal diagonal of elasticity matrix (elastic coupling of movements), eventually by the dissipation coefficients from the viscous damping matrix if they are significant;

**c)** comparing the values of the pulsations/frequencies from the **table 2** and **table 3**, we can say that the movements inside the subsystems almost uncoupled on the "directions"  $(X, Y, Z, \varphi_x, \varphi_y, \varphi_z)$ ; also the values very small or very big of the eigenvalues can explain the quasidecoupling of the movements of the subsystems;

**d)** analyzing the values from **table 3**, we can find a group of three natural frequencies in the domain  $1,1 \div 1,2$  Hz and another one in the domain  $15,8 \div 17,1$  Hz; this grouping of frequencies and the big differences between the values of domains' limits can be explained by the significant differences between the bearings stiffness on vertical axis **Cz** (compression effort) and on horizontal plane xCy (shear efforts);

#### ACKNOWLEDGEMENT

The author acknowledges the support of the people from the Vibration and Acoustic Laboratory of the Research Institute for Construction Equipment and Technology ICECON S.A. Bucharest, with which the instrumental tests have been performed in the site of Transylvania highway.

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